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# Reliability sensitivity analysis based on multi-hyperplane combination method

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**Abstract**

For a nonlinear limit state function, the first-order reliability method (FORM) may cause large errors in the computation of not only the reliability index or failure probability but also the reliability sensitivity. In order to obtain more the accurate results of the reliability sensitivity analysis, a number of hyperplanes are built near the design point by first-order Tayler series expansion, which replace the known nonlinear limit state hypersurface, and an equivalent computational method is utilized to construct an equivalent hyperplane of the obtained hyperplanes. And the reliability sensitivities can be estimated more accurately by the derived equations based on the equivalent hyperplane. An example shows that the method is effective and feasible.


*Keywords:* Reliability sensitivity; Failure probability; Nonlinearity; Multi-hyperplane combination method; Equivalence

## 1. Introduction

Reliability sensitivity analysis plays an important role in reliability design, reliability-based optimization design and reliability-based robust design. Reliability sensitivities are useful to quantify the distribution parameters such as mean and standard deviation, and represent that the distribution parameters influence the reliability to some degree. The reliability sensitivities are often expressed as the sensitivity of the computed failure probability to changes in the distribution parameters.

Among the methods available for the parametric sensitivity analysis, the method based on FORM is a fundamental and widely used approach due to the high computational efficiency and acceptable accuracy of FORM [1–3]. Obviously, FORM is to replace a certain limit state hypersurface with a hyperplane and makes it easy to calculate the failure or safety probability. Because of this, FORM may overestimate or underestimate the reliability when the hypersurface is nonlinear near the design point. The higher the curvature of the limit state hypersurface at the design point is, the bigger the computational error of the reliability is. Just as FORM usually overestimates or underestimates the reliability, it may underestimate or overestimate the degree to which the distribution parameters influence the reliability. Therefore, the parametric sensitivity analysis based on FORM may be unsatisfactory or unacceptable.

Compared with FORM, SORM (second-order reliability method) can compute the reliability more accurately. But because of the difficulty in computing the reliability, the parametric sensitivity analysis based on SORM was seldom discussed [4]. MCSM (Monte Carlo simulation method) can be used for the sensitivity analysis [5–8]. The simulated

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results of the sensitivity analysis by MCSM are quite accurate, but one main shortcoming of MCSM is that the computational amount is too large.

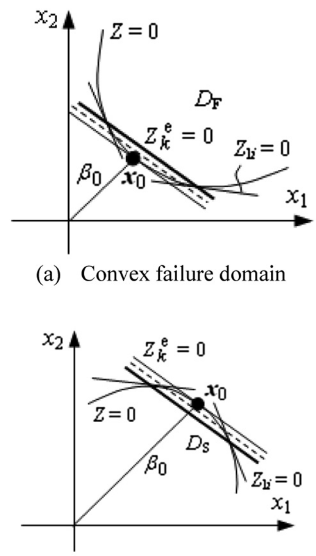
MHCM (multi-hyperplane combination method) is an alternative to improve the calculation accuracy of the reliability for a nonlinear limit state function [9–11]. In MHCM, some hyperplanes are utilized to replace the limit state hypersurface. Feng [9] first studied MHCM for obtaining much more accurate computational result of reliability. Mahadevan et al. [10] used the idea of MHCM for a little more complex reliability calculation with multiple extreme points, but the handling method in Ref. [10] is rough and the calculation accuracy of the reliability is not high enough. Shin et al. [11] introduced a progressive importance sampling method based on multi-hyperplane combination and the importance sampling for the reliability analysis of the nonlinear performance function with multiple design points. Lu et al. [12] studied how to obtain some hyperplanes and how to compute reliability easily when MHCM was used. The computed results of MHCM are different when a different number of the hyperplanes and the different hyperplanes are selected. And the different way to calculate the reliability of the system consisting of the obtained hyperplanes may lead to the results with different calculation accuracies. Anyhow, compared to FORM, MHCM can improve the calculation accuracy of the reliability greatly. Compared to MCSM, MHCM requires less computational amount.

As MHCM can get more accurate computational result of the reliability, it can obtain more accurate results of the reliability sensitivity analysis compared to FORM. In this paper, some equations of computing the sensitivities for the calculated failure probability were derived according to MHCM, and an example was given to show that the method of the sensitivity analysis based on MHCM is effective and feasible in improving the calculation accuracy of the sensitivity analysis.

## 2. Basic models of MHCM

Assuming that $x_1$, $x_2$, $x_3$, …, $x_n$ are basic random variables which are independently subjected to standard normal distribution and $\boldsymbol{x} = [x_1, x_2, x_3, \ldots, x_n]$ is a $1 \times n$ order matrix, a nonlinear limit state function can be expressed by $Z = g(\mathbf{x})$, and the limit state hypersurface can be expressed by $Z = g(\mathbf{x}) = 0$. To obtain a number of hyperplanes to replace the limit state hypersurface, the same number of points on the hypersurface and near the design point is obtained by extrapolation method and one-dimensional iterative search method [12]. Then, a number of hyperplanes can be built at the obtained points by the first-order Tayler series expansion.

In Fig. 1, the hyperplane (for two dimensions, namely, two random variables, the hypersurface is reduced to a line) at the design point $\mathbf{x}_0$ is acquired by the first-order Tayler series expansion of the nonlinear limit state function, which is called as a main hyperplane, denoted by $Z_{l0} = 0$. The other hyperplanes at the above obtained points except $\mathbf{x}_0$ are called as sub-hyperplanes, denoted by $Z_{li} = 0$ ($i = 1, 2, \ldots, k$).



(a) Convex failure domain

(b) Convex safety domain

Fig. 1. Reliability computated by MHCM.

As shown in Fig. 1, the dotted lines express the course of equivalent calculation, and each dotted line is expressed as a main equivalent approximate line by an equivalent calculation. The last main equivalent approximate line is expressed by a thick solid line.

The safety domain $D_{Si}$ of each hyperplane (main hyperplane or sub-hyperplanes) can be denoted by

$$D_{Si} = \{\mathbf{x} | Z_{li} > 0\} (i = 0, 1, 2, \cdots, k) \tag{1}$$

Because the limit state function $Z_{li}$ is linear, the safety probability $P_{Si}$ in which the values of $\mathbf{x}$ are located in $D_{Si}$ can be given accurately by

$$P_{Si} = P(D_{Si}) = \Phi(\beta_i) \tag{2}$$

where $\Phi(\bullet)$ is the accumulative integral function of standard normal distribution; and $\beta_i$ is the reliability index of the linear limit state function $Z_{li}$.

In Fig. 1(a), when the original nonlinear failure domain is a local convex set, the original nonlinear safety domain $D_S$ can be expressed approximately by the set union of all $D_{Si}$ as follows

$$D_S \approx \bigcup_{i=0}^{k} D_{Si} \tag{3}$$

According to the basic model of system reliability, the system consisting of the above hyperplanes is a parallel one. The safety probability $P_S$ in which the values of $\boldsymbol{x}$ are located in $D_S$ can be given by

$$P_S = P(D_S) \approx P\left(\bigcup_{i=0}^{k} D_{Si}\right) \tag{4}$$

It is a little complicated to calculate $P_S$ directly using Eq. (4), because the operation of set union should be transformed into that of set intersection.

Based on the duality of system reliability, when the original nonlinear failure domain is a convex set, the nonlinear failure domain can also be expressed by set intersection of failure domain $D_{Fi}$ of each hyperplane, as shown in Fig. 1(a). $D_{Fi}$ is written as

$$D_{Fi} = \{\boldsymbol{x} | Z_{li} \le 0\} (i = 0, 1, 2, \cdots, k) \tag{5}$$

It is well known that not only the failure domain but also the safety domain of each hyperplane is a linear domain. The failure probability $P_{Fi}$ in which the values of $\mathbf{x}$ are located in $D_{Fi}$ can be obtained

$$P_{Fi} = P(D_{Fi}) = \Phi(-\beta_i) \tag{6}$$

The original nonlinear failure domain $D_F$ can be expressed approximately by the set intersection of all $D_{Fi}$ as follows

$$D_F \approx \bigcap_{i=0}^{k} D_{Fi} \tag{7}$$

The failure probability $P_F$ in which the values of $\boldsymbol{x}$ are located in $D_F$ can be given by

$$P_F = P(D_F) \approx P\left(\bigcap_{i=0}^{k} D_{Fi}\right) \tag{8}$$

Assuming that $D$ is the full probability space, we have $D_F \cup D_S = D$, $D_F \cap D_S = \phi$, and $P_F + P_S = 1$. So, $P_F$ or$P_S$ can be computed to obtain the reliability. From the above discussion, it is known that, when the original nonlinear failure domain is the convex set, it is much more easy to compute the failure probability $P_F$ of the original nonlinear limit state function, that is to say, to compute the joint probability of the failure domains of the obtained hyperplanes.

Accordingly, as shown in Fig. 1(b), when the original nonlinear safety domain is a local convex set, it is easy to compute the safety probability of the original nonlinear safety domain by using the safety domain of each hyperplane. The safety domain is replaced approximately by the set intersection of the safety domains of the above obtained hyperplanes, namely

$$D_S \approx \bigcap_{i=0}^{k} D_{Si} \tag{9}$$

$P_S$ can be obtained as

$$P_S = P(D_S) \approx P\left(\bigcap_{i=0}^{k} D_{Si}\right) \tag{10}$$

In fact, the computation of the safety probability is transformed into that of series system consisting of the above hyperplanes.

Therefore, computing $P_F$ or $P_S$ depends on whether the failure domain or the safety domain is a convex domain. It is much more easy to calculate the probability of a convex domain.

If neither the failure domain nor the safety domain of the original nonlinear limit state function is a convex domain, the system consisting of the above hyperplanes is neither a parallel one nor a series one, and it is much more complicated to compute the system reliability. In the paper, the case is not discussed for the time being.

## 3. An algorithm of MHCM

It is not difficult to calculate $P_{Fi}$ or $P_{Si}$ of each hyperplane using Eq. (6) or Eq. (2), but it is comparatively difficult to compute the joint probability $P_F$ or $P_S$ of all the obtained hyperplanes by using Eq. (8) or Eq. (10). In the paper, the failure or safety probability of the system consisting of the obtained hyperplanes is gotten by a method, called as a sequential equivalent computing algorithm [12].

After $k+1$ hyperplanes are acquired, they are arranged in the increasing order of the distances between the origin and the obtained points. The arranged hyperplanes are $Z_{l0} = 0$, $Z_{l1} = 0$, …, $Z_{lk} = 0$, where $Z_{l0} = 0$ is the obtained main hyperplane, and $Z_{l1} = 0$, …, $Z_{lk} = 0$ are the obtained sub-hyperplanes.

According to the sequential equivalent computing algorithm, the first equivalence computation is obtained by the main hyperplane $Z_{l0} = 0$ and the sub-hyperplane $Z_{l1} = 0$. Therefore, the first equivalent hyperplane $Z_1^e = 0$ can be obtained. The equivalent conditions are as follow: (1) the obtained equivalent hyperplane is parallel to the main hyperplane; and (2) the reliability of the equivalent hyperplane is equal to that of system consisting of the two hyperplanes.

The $i$th ($i = 2, 3, \ldots, k$) equivalence hyperplane $Z_i^e = 0$ is gotten using the $(i-1)$th equivalent hyperplane $Z_{i-1}^e = 0$ and the $i$th sub-hyperplane $Z_{li} = 0$. Therefore, the $k$th equivalence hyperplane or the last equivalence hyperplane $Z_k^e = 0$ is gotten by using $Z_{k-1}^e = 0$ and the last sub-hyperplane $Z_{lk} = 0$.

For the convex failure domain, the method of obtaining any equivalent hyperplane is as follows.

The equivalent failure probability of a parallel system consisting of two hyperplanes is given by

$$P_{Fi}^e = \Phi_2\left(-\beta_{i-1}^e, -\beta_i; \rho_{0,\,i}\right)\left(i = 1, 2, \cdots, k\right) \tag{11}$$

where $P_{Fi}^e$ is the equivalent failure probability of the system consisting of $Z_{i-1}^e = 0$ and $Z_{li} = 0$ (if $i = 1$, $Z_{i-1}^e = 0$ is $Z_{l0} = 0$); $\Phi_2(\bullet)$ is the accumulative integral function of two-dimensional standard normal distribution;$\beta_{i-1}^e$ is the $(i-1)$th equivalent reliability index; $\beta_i$ is the reliability index of the sub-hyperplane $Z_{li} = 0$; and $\rho_{0,i}$ is the correlation coefficient between $Z_{l0} = 0$ (or $Z_{i-1}^e = 0$) and $Z_{li} = 0$.

The $i$th equivalent reliability index $\beta_i^e$ is

$$\beta_i^e = \Phi^{-1}\left(1 - P_{Fi}^e\right) \tag{12}$$

where $\Phi^{-1}(\bullet)$ is the inverse function of accumulative integral function of standard normal distribution.

The $i$th equivalent hyperplane $Z_i^e = 0$ is

$$Z_i^e = \boldsymbol{\alpha}^e \boldsymbol{x}^T + \beta_i^e = 0 \tag{13}$$

where $\boldsymbol{\alpha}^e$ is a $1 \times n$ order matrix, which is the same as $\boldsymbol{\alpha}_0$ of the main hyperplane, because $Z_i^e = 0$ is parallel to the main hyperplane $Z_{l0} = 0$. $\boldsymbol{\alpha}^e$ is denoted by

$$\boldsymbol{\alpha}^{\mathrm{e}} = \boldsymbol{\alpha}_0 = [\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n] \tag{14}$$

Obviously, it is easy to know that

$$\sum_{i=1}^{n} \alpha_i^2 = 1 \tag{15}$$

The process of equivalence is shown in Fig. 1(a).

Especially, the comprehensive failure probability $P'_{\mathrm{F}}$ of the system consisting of the obtained hyperplanes $Z_{l0} = 0, Z_{l1} = 0, \ldots, Z_{lk} = 0$ can be estimated by

$$P'_{\mathrm{F}} \approx P^{\mathrm{e}}_{\mathrm{F}k} = \Phi_2\left(-\beta^{\mathrm{e}}_{k-1}, -\beta_k; \rho_{0,k}\right) \tag{16}$$

The failure probability $P_{\mathrm{F}}$ of the original limit state function can be estimated by $P_{\mathrm{F}} \approx P'_{\mathrm{F}}$. And the reliability index $\beta$ can be given by $\beta = \Phi^{-1}(1 - P_{\mathrm{F}})$.

Accordingly, as shown in Fig. 1(b), for the convex safety domain, the method of obtaining any equivalent hyperplane is as follows.

The equivalent reliability of a series system consisting of two hyperplanes is given by

$$P^{\mathrm{e}}_{\mathrm{S}i} = \Phi_2\left(\beta^{\mathrm{e}}_{i-1}, \beta_i; \rho_{0,\,i}\right)\left(i = 1, 2, \cdots, k\right) \tag{17}$$

The $i$th equivalent reliability index is

$$\beta^{\mathrm{e}}_i = \Phi^{-1}\left(P^{\mathrm{e}}_{\mathrm{S}i}\right) \tag{18}$$

The $i$th equivalent hyperplane $Z^{\mathrm{e}}_i = 0$ can also be expressed by Eq. (13).

$P_{\mathrm{S}}$ and $P_{\mathrm{F}}$ of the original limit state function can be estimated by

$$P_{\mathrm{S}} \approx P^{\mathrm{e}}_{\mathrm{S}k} = \Phi_2\left(\beta^{\mathrm{e}}_{k-1}, \beta_k; \rho_{0,k}\right) \tag{19}$$

$$P_{\mathrm{F}} = 1 - P_{\mathrm{S}} \approx 1 - P^{\mathrm{e}}_{\mathrm{S}k} = 1 - \Phi_2\left(\beta^{\mathrm{e}}_{k-1}, \beta_k; \rho_{0,k}\right) \tag{20}$$

## 4. Sensitivity analysis using MHCM

From the section above, it is known that the final equivalent hyperplane can be expressed by

$$Z^{\mathrm{e}}_k = \boldsymbol{\alpha}^{\mathrm{e}}\boldsymbol{x}^{\mathrm{T}} + \beta^{\mathrm{e}}_k = 0 \tag{21}$$

When the random variables are not standard normal variables, they should be transformed into standard normal random variables. Assuming that a random variable $y_i(i = 1, 2, 3, \cdots, n)$ is a normal variable, and its mean value and standard deviation are $\mu_i$ and $\sigma_i$, respectively. Let two $1 \times n$ order matrixes be

$$\boldsymbol{y} = [y_1, y_2, y_3, \cdots\cdots, y_n] \tag{22}$$

$$\boldsymbol{\mu} = [\mu_1, \mu_2, \mu_3, \cdots\cdots, \mu_n] \tag{23}$$

And, $\boldsymbol{\sigma}$ is an $n \times n$ order diagonal matrix, of which the diagonal elements are $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$ respectively.

Eq. (24) can be introduced.

$$\boldsymbol{x} = (\boldsymbol{y} - \boldsymbol{\mu})/\boldsymbol{\sigma} \tag{24}$$

Substituting Eq. (24) into Eq. (21), we have

$$Z^{\mathrm{e}}_k = (\boldsymbol{\alpha}^{\mathrm{e}}/\boldsymbol{\sigma})\boldsymbol{y}^{\mathrm{T}} + \beta^{\mathrm{e}}_k - (\boldsymbol{\alpha}^{\mathrm{e}}/\boldsymbol{\sigma})\boldsymbol{\mu}^{\mathrm{T}} = 0 \tag{25}$$

Let $\boldsymbol{c} = [c_1, c_2, c_3, \cdots\cdots, c_n] = (\boldsymbol{\alpha}^{\mathrm{e}}/\boldsymbol{\sigma})$ be a $1 \times n$ order matrix, and $c_0 = \beta^{\mathrm{e}}_k - (\boldsymbol{\alpha}^{\mathrm{e}}/\boldsymbol{\sigma})\boldsymbol{\mu}^{\mathrm{T}}$, it is easy to know that $c_0$ is a constant, and $c_i$ can be given by

$$c_i = \alpha_i/\sigma_i \tag{26}$$

Eq. (21) can be rewritten as

$$Z^{\mathrm{e}}_k = \mathbf{c}\mathbf{y}^{\mathrm{T}} + c_0 = 0 \tag{27}$$

Based on Eq. (27), $\mu_i$ and $\sigma_i$ can be used to obtain the equivalent mean value $u^{\mathrm{e}}_k$, the standard deviation $\sigma^{\mathrm{e}}_k$, the reliability index $\beta^{\mathrm{e}}_k$ and the failure probability $P^{\mathrm{e}}_{\mathrm{F}k}$ of the equivalent hyperplane

$$u^{\mathrm{e}}_k = \boldsymbol{c}\boldsymbol{\mu}^{\mathrm{T}} + c_0 = \sum_{i=1}^{n} c_i\mu_i + c_0 \tag{28}$$

$$\sigma^{\mathrm{e}}_k = \sqrt{\sum_{i=1}^{n} c_i^2\sigma_i^2} \tag{29}$$

$$\beta^{\mathrm{e}}_k = \mu^{\mathrm{e}}_k/\sigma^{\mathrm{e}}_k \tag{30}$$

$$P^{\mathrm{e}}_{\mathrm{F}k} = \Phi\left(-\beta^{\mathrm{e}}_k\right) = \Phi\left(-\mu^{\mathrm{e}}_k/\sigma^{\mathrm{e}}_k\right) \tag{31}$$

From Eqs. (15) and (26), we know that the sensitivities of $\mu_i$ and $\sigma_i(i = 1, 2, 3, \cdots, n)$ for the computed failure probability $P^{\mathrm{e}}_{\mathrm{F}k}$ can be gotten by

$$\frac{\partial P^{\mathrm{e}}_{\mathrm{F}k}}{\partial\mu_i} = \frac{\partial P^{\mathrm{e}}_{\mathrm{F}k}}{\partial\beta^{\mathrm{e}}_k} \times \frac{\partial\beta^{\mathrm{e}}_k}{\partial\mu^{\mathrm{e}}_k} \times \frac{\partial\mu^{\mathrm{e}}_k}{\partial\mu_i} = -\frac{\alpha_i}{\sigma_i}\Phi(\beta^{\mathrm{e}}_k) \tag{32}$$

$$\frac{\partial P^{\mathrm{e}}_{\mathrm{F}k}}{\partial\sigma_i} = \frac{\partial P^{\mathrm{e}}_{\mathrm{F}k}}{\partial\beta^{\mathrm{e}}_k} \times \frac{\partial\beta^{\mathrm{e}}_k}{\partial\sigma^{\mathrm{e}}_k} \times \frac{\partial\sigma^{\mathrm{e}}_k}{\partial\sigma_i} = \frac{\beta^{\mathrm{e}}_k\alpha_i^2}{\sigma_i}\Phi(\beta^{\mathrm{e}}_k) \tag{33}$$

where $\Phi(\bullet)$ is the probability density function of the standard normal distribution.

Referring to Fig. 1, using Eqs. (32) and (33) can come to the conclusion that the absolute values of the sensitivities computed based on MHCM are usually smaller than those computed based on FORM when the failure domain is a convex one. The reason is that the reliability index computed by using MHCM is bigger than that computed by using FORM. When the safety domain is a convex one, the opposite conclusion is obtained.

## 5. An example

A pull rod with equal diameter is subjected to a constant load $F = 1.5e + 5N$. The mean values and standard deviations of the diameter $d$ and yield limit $r$ are $\mu_d = 32$ mm, $\sigma_d = 3$ mm, and $\mu_r = 295$ MPa, $\sigma_r = 25$ MPa, respectively. Let's compute the failure probability and parametric sensitivities of the pull rod.

Assuming $\boldsymbol{y} = [y_1, y_2] = [d, r]$, according to the load limit, the limit state function can be expressed by

$$Z = (\pi/4)y_1^2y_2 - 150\ 000 \tag{34}$$

Let $x_1$, $x_2$ ~ $N(0,1)$, we have

Table 1
Parameters of lines.

| No. | Obtained line | | Equivalent line | |
|---|---|---|---|---|
| 0 | $\boldsymbol{\alpha}$ | [0.929 493 6, 0.368 838 3] | – | |
| | $\beta$ | 2.044 659 4 | | |
| 1 | $\boldsymbol{\alpha}$ | [0.945 056 5, 0.326 907 0] | $\boldsymbol{\alpha}^{\mathrm{e}}$ | $\boldsymbol{\alpha}_0$ |
| | $\beta$ | 2.067 100 8 | $\beta^{\mathrm{e}}$ | 2.036 127 7 |
| 2 | $\boldsymbol{\alpha}$ | [0.912 009 7, 0.410 168 7] | $\boldsymbol{\alpha}^{\mathrm{e}}$ | $\boldsymbol{\alpha}_0$ |
| | $\beta$ | 2.064 697 0 | $\beta^{\mathrm{e}}$ | 2.029 243 8 |
| 3 | $\boldsymbol{\alpha}$ | [0.951 323 3, 0.308 194 6] | $\boldsymbol{\alpha}^{\mathrm{e}}$ | $\boldsymbol{\alpha}_0$ |
| | $\beta$ | 2.092 705 7 | $\beta^{\mathrm{e}}$ | 2.024 068 3 |
| 4 | $\boldsymbol{\alpha}$ | [0.903 584 6, 0.428 409 7] | $\boldsymbol{\alpha}^{\mathrm{e}}$ | $\boldsymbol{\alpha}_0$ |
| | $\beta$ | 2.085 654 2 | $\beta^{\mathrm{e}}$ | 2.018 474 6 |

$$[y_1, y_2] = [3x_1 + 32, 25x_2 + 295] \tag{35}$$

Substituting Eq. (35) into Eq. (34) yields

$$Z = (\pi/4)(3x_1 + 32)^2(25x_2 + 295) - 150\ 000 \tag{36}$$

FORM is used to obtain the reliability index $\beta = 2.044\ 659\ 4$ and the failure probability $P_{\mathrm{F}} = 0.020\ 444\ 2$. The limit state function expressed by Eq. (36) is nonlinear so that the failure and safety domains are nonlinear, too. Because of this, the failure probability computed using FORM may be not accurate enough. It cannot ensure that the sensitivities of FORM failure probability are accurate enough.

In this reliability problem, the most probable failure domain can be thought to be only located near the design point, and the safety domain near the design point is a convex one. Clearly, for $n = 2$, the hyperplanes are reduced to the lines. In order to illustrate the effectiveness of the method proposed in the paper, one main line $Z_{10} = 0$ and four sub-lines $Z_{11} = 0$, $Z_{12} = 0$, $Z_{13} = 0$, $Z_{14} = 0$ are gotten to replace the limit state curve. It is not difficult to know that the five lines consist of a series system.

According to the five lines, four corresponding reliability indexes can be computed by using of Eqs. (17) and (18). Also, four equivalent lines can be obtained by using Eq. (13), and they are parallel to the main line. The parameters $\boldsymbol{\sigma}$ and $\beta$ of the five obtained lines and four equivalent lines are listed in Table 1.

It is known that the original limit state curve is replaced equivalently by the last equivalent line $Z_4^{\mathrm{e}} = 0$. The computational results of $\beta$, $P_{\mathrm{F}}$ and the errors of $P_{\mathrm{F}}$ are given in Table 2. The simulation number of MCSM is $10^7$

It can be seen from Table 2 that $\beta$ and $P_{\mathrm{F}}$ computed by MHCM is very close to those obtained by MCSM. Compared with MCSM, the error of $P_{\mathrm{F}}$ computed by MHCM or the last equivalent line is only $-0.083\%$. The error is less than that computed by SORM, and far less than that computed by FORM.

Table 2
Comparisons of failure probabilities.

| Method | $\beta$ | $P_{\mathrm{F}}$ | Error/% |
|---|---|---|---|
| FORM | 2.044 659 4 | 0.020 444 2 | −6.172 |
| SORM | 2.025 393 8 | 0.021 413 5 | −1.723 |
| MHCM | 2.018 474 6 | 0.021 770 9 | −0.083 |
| MCSM | 2.018 127 4 | 0.021 789 0 | – |

Table 3
Comparisons of sensitivities.

| Method | FORM (error)/% | MHCM (error)/% | MCSM |
|---|---|---|---|
| $\mu_d$ | −0.015 283 5 (−3.961) | −0.016 118 5 (1.286) | −0.015 913 9 |
| $\sigma_d$ | 0.029 046 2 (−2.645) | 0.030 240 9 (1.560) | 0.029 835 3 |
| $\mu_r$ | −0.000 727 8 (−5.098) | −0.000 767 5 (0.078) | −0.000 766 9 |
| $\sigma_r$ | 0.000 548 8 (−17.818) | 0.000 571 4 (−14.436) | 0.000 667 8 |

In the example, because FORM is to replace the convex nonlinear safety domain with a linear safety domain which is determined by a tangent line of the limit state curve through the design point, the actually calculated safety domain of FORM is far greater than safety domain of the original limit state function, and FORM overestimates reliability. So, the safety probability computed by FORM is far bigger than the true safety probability. That is, the failure probability computed by FORM is far smaller than the true failure probability (see the last row of Table 2). It can be seen from Fig. 1(b) that the safety domain calculated by MHCM is a little greater than the original safety domain, so the failure probability computed by MHCM is a little smaller than the true failure probability.

MHCM can be used to obtain more accurate computational results of not only the failure probability but also sensitivity analysis. The computed sensitivities and the absolute value errors of the computed sensitivities are listed in Table 3. The simulation number of MCSM is also $10^7$.

From Table 3, some conclusions can be drawn as follows.

(1) The absolute values of the sensitivities by MHCM are bigger than those by FORM.
(2) Compared with the sensitivities computed by FORM, the sensitivities computed by MHCM are closer to the sensitivities simulated by MCSM
(3) Although FORM overestimates the reliability (see Table 2, the reliability index computed by using FORM is bigger than the true value, or the failure probability computed by using FORM is smaller the true value), it underestimates the degree to which the distribution parameters influence the reliability.
(4) The absolute values of the sensitivities computed by MHCM may be bigger or smaller than the ones simulated by MCSM. The reason is that the algorithm of MHCM in the paper may cause the errors that can not be easy to know and control.
(5) The sensitivities of the parameters $\mu_d$, $\sigma_d$ and $\mu_r$ computed by MHCM are much more easier to be close to the sensitivities estimated by MCSM. In the example, the five lines can be used to make the failure probability relatively accurate and the sensitivities are close to those by MCSM, but it can not make all the sensitivities accurate enough.

## 6. Conclusions

If the failure probability of a nonlinear limit state function cannot be computed accurately by a method, the sensitivities for the failure probability cannot be computed accurately by the method either. In the case of the nonlinear limit state function, using FORM to analyze the sensitivities almost certainly underestimates or overestimates the degree to which the parameters of random variables influence the reliability. By replacing the original limit state hypersurface with some hyperplanes, MHCM can improve the calculation accuracy of the reliability and the sensitivities, and makes all the computational results closer to their corresponding true values.

To obtain much more accurate computational results of the reliability and the sensitivities, more hyperplanes may be selected if necessary. In practical applications, not so many hyperplanes are required to get enough accurate results of computing the reliability and the sensitivities. For a certain nonlinear limit state function with single design point, $2n + 1$ hyperplanes are a good choice. It is undoubted that how to select the number of hyperplanes and how to satisfy the requirement of the given accuracy in calculating the reliability and the sensitivities should be made in the further research.

## References

[1] Karamchandani A, Cornell C. Sensitivity estimation within first and second order reliability methods. Struct Saf 1991;11(2):95–107.

[2] Bjerager P, Krenk S. Parametric sensitivity in first order reliability analyses. J Eng Mech 1989;115(7):1577–82.

[3] Jin YJ, Zhang YM, Zhang YL. Analysis of reliability and reliability sensitivity for machine components by mean-value first order saddlepoint approximation. J Mech Eng 2009;45(12):102–7 [In Chinese].

[4] Breitung K. Asymptotic approximations for multinormal integrals. J Eng Mech 1984;110(3):357–66.

[5] Melchers RE, Ahammed M. A fast approximate method for parameter sensitivity estimation in Monte Carlo structural reliability. Comput Struct 2004;82(1):55–61.

[6] Wu YT, Sitakanta M. Variable screening and ranking using sampling-based sensitivity measures. Reliab Eng Syst Saf 2006;91(6):634–47.

[7] Zhang F, Lu ZZ. An adaptive importance sampling method for estimation of reliability sensitivity. Eng Mech 2008;25(4):80–4 [In Chinese].

[8] Song SF, Lu ZZ. The reliability sensitivity analysis based on saddle point approximation and its improved method. Chin J Theor Appl Mech 2011;43(1):162–8 [In Chinese].

[9] Feng YS. The computation of failure probability for nonlinear safety margin equation. Reliab Eng Syst Saf 1990;27(3):323–31.

[10] Mahadevan S, Shi P. Multiple linearization method for nonlinear reliability analysis. J Eng Mech 2001;127(11):1165–73.

[11] Shin Youngwon, (Justin) Wu Y-T. Progressive importance sampling approach for reliability analysis. In: 47th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Newport, Rhode Island; 2014. p. 1–8.

[12] Lu HT, Dong YG. Study of multiple-plane combination approximation method for structural reliability analysis. Chin J Comput Mech 2013;30(3):343–8 [In Chinese].